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On the Strong Cesaro Summability of Double Orthogonal Expansions

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Denoting by $\sigma_{m,n}^{(\alpha,\beta)}$ the (m, n)th Cesaro mean of the double series $\sum a_{i,k}$, we say that the double series is strongly Cesaro summable with parameters $\alpha, \beta \geqslant 0$ and index $\lambda > 0$—or summable $[C, (\alpha, \beta)]_\lambda$—to the sum s if in the cases

(i) $\alpha, \beta > 0$, $(m+1)^{-1}(n+1)^{-1} \sum_{i=0}^{m} \sum_{k=0}^{n} |\sigma_{i,k}^{(\alpha-1,\beta-1)} - s|^\lambda = o(1)$.

(ii) α or β or both are zero

$$(m+1)^{-1}(n+1)^{-1} \sum_{i=0}^{m} \sum_{k=0}^{n} |\Delta_{11}((i+1)(k+1)\sigma_{i,k}^{(\alpha,\beta)}) - s|^\lambda = o(1),$$

where $\Delta_{11}\omega_{i,k} = \omega_{i,k} - \omega_{i-1,k} - \omega_{i,k-1} + \omega_{i-1,k-1}$ and $\omega_{m,n} = o(1)$ means that $\omega_{m,n} \to 0$ in Pringsheim's sense and there exists a constant C such that for any pair m, n of natural numbers $|\omega_{m,n}| \leqslant C$ holds. This definition for single series was introduced in the case (i) by J. M. Hyslop (*Proc. Glasgow Math. Assoc.* **1** (1952), 16–20) and in the case (ii) by N. Tanovic-Miller, (*Acta Math. Hungar.* **42** (1983), 35–43). The case $\alpha = \beta = 0$ is the strong convergence, where $\sigma_{i,k}^{(0,0)} = s_{i,k}$ the (i, k)th rectangular partial sum of series. The double orthogonal series $\sum c_{i,k}\varphi_{i,k}(x, y)$ with the coefficient condition $\sum c_{i,k}^2 < \infty$ is an orthogonal expansion of a square-integrable function $f(x, y)$. Exchanging the coefficient condition by a stronger one F. Móricz (*Stud. Math.* **81** (1985), 79–94) proved a theorem for the $[C, (\alpha, \beta)]_2$; $\alpha, \beta > \frac{1}{2}$ summability of orthogonal series to $f(x, y)$ almost everywhere. Now we extend the investigation for the wider range of parameters and indices. © 1989 Academic Press, Inc.

1. The Concept of the Summability in Pringsheim's Sense with a Bound

The double series

$$\sum_{i,k=0}^{\infty} a_{i,k} \tag{1.1}$$

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is said to be Cesaro summable—or summable $C_{\alpha,\beta}$—to s, with parameters $\alpha, \beta > -1$ if $\sigma_{m,n}^{(\alpha,\beta)} \to s$ in Pringsheim's sense, where

$$\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha)} A_{n-k}^{(\beta)} a_{i,k}, \qquad m, n = 0, 1, 2, \ldots \tag{1.2}$$

and

$$A_0^{(\alpha)} = 1; \qquad A_n^{(\alpha)} = \frac{(1+\alpha)(2+\alpha)\cdots(n+\alpha)}{n!}, \qquad n = 1, 2, 3, \ldots$$

(see, e.g., [3, p. 185–186]). $\sigma_{m,n}^{(\alpha,\beta)}$ is the (m, n)th rectangular $(C, (\alpha, \beta))$ mean of the sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$, while $s_{m,n} = \sigma_{m,n}^{(0,0)}$ is the (m, n)th rectangular partial sum of the series (1.1).

Considering a one-dimensional series $\sum_{i=0}^{\infty} a_i$ and writing

$$a_{i,k} = \begin{cases} a_i, & \text{if } \; k = 0, \; i = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases} \tag{1.3}$$

the definition of summability $C_{\alpha,\beta}$ gives back the well-known definition of summability (C, α), independent of parameter β. On the other hand the method $C_{\alpha,\beta}$ has a remarkable deviation from the method (C, α), namely, while for one-dimensional series the method $(C, \alpha + \delta)$, $\delta > 0$, is stronger than method (C, α), the implication $C_{\alpha+\delta,\beta+\gamma} \Leftarrow C_{\alpha,\beta}$, $(\delta, \gamma > 0)$, holds under certain conditions, only. The exact condition (see [3, p. 187, Th. 2]) is complicated for the applications. Now we introduce a restricted but more comfortable Cesaro method, which will be useful for the investigations of double orthogonal series.

First of all, we say that the double sequence $\{\omega_{m,n}\}_{m,n=0}^{\infty}$ converges to ω, in Prinsheim's sense with a bound if $\omega_{m,n} \to \omega$ in Pringsheim's sense, and in addition, it is bounded. More exactly, this means that for any positive ε there is a number κ, such that if $m, n > \kappa$ then $|\omega_{m,n} - \omega| < \varepsilon$ and in addition, there exists a constant C, such that $|\omega_{m,n} - \omega| \leqslant C$, for any m and n.

In this paper

$$\omega_{m,n} = o(1), \qquad \text{as} \quad \min(m, n) \to \infty, \tag{1.4}$$

means that $\{\omega_{m,n}\}_{m,n=0}^{\infty}$ converges to O, in Pringsheim's sense with a bound, while

$$\omega_{m,n} = O(1)$$

means that the sequence $\{\omega_{m,n}\}_{m,n=0}^{\infty}$ is bounded. Now we have

Property 1.5 [9, Lemma 1.11]. *Let be* $\bar{\alpha}, \bar{\beta} > -1$ *and* $\delta, \gamma > 0$. *If* (1.4) *is satisfied, then*

$$\frac{1}{(m+1)^{\bar{\alpha}+\delta}} \sum_{i=0}^{m} (m-i+1)^{\delta-1} (i+1)^{\bar{\alpha}} \omega_{i,n} = o(1), \qquad \textit{as} \quad \min(m,n) \to \infty, \tag{1.5.1}$$

$$\frac{1}{(n+1)^{\bar{\beta}+\gamma}} \sum_{k=0}^{n} (n-k+1)^{\gamma-1} (k+1)^{\bar{\beta}} \omega_{m,k} = o(1), \qquad \textit{as} \quad \min(m,n) \to \infty, \tag{1.5.2}$$

and

$$\frac{1}{(m+1)^{\bar{\alpha}+\delta} (n+1)^{\bar{\beta}+\gamma}} \sum_{i=0}^{m} \sum_{k=0}^{n} (m-i+1)^{\delta-1} (n-k+1)^{\gamma-1} \times (i+1)^{\bar{\alpha}} (k+1)^{\bar{\beta}} \omega_{i,k} = o(1), \tag{1.5.3}$$

as $\min(m,n) \to \infty$, *hold.*

The double series (1.1) is said to be Cesaro summable in Pringsheim's sense with a bound—or summable $(C, (\alpha, \beta))$—to s, with parameters $\alpha, \beta > -1$ if $\sigma_{m,n}^{(\alpha,\beta)} - s = o(1)$. Of course, the $(C, (0,0))$ summability denotes that the series (1.1) is convergent in Pringsheim's sense with a bound. Using (1.3) the definition of summability $(C, (\alpha, \beta))$ gives back the definition of summability (C, α), because in the one-dimensional case the convergence implies the boundedness. Obviously, we see

Property 1.6. *If* $\sigma_{m,n}^{(\alpha,\beta)} = O(1)$, *then* $C_{\alpha,\beta} \Rightarrow (C, (\alpha, \beta))$.

Moreover we have

Property 1.7 (see [3, p. 189, Th. 5] or [9, Corollary 1.12]). *If* $\alpha, \beta > -1$ *and* $\delta, \gamma \geqslant 0$ *then* $(C, (\alpha+\delta, \beta+\gamma)) \Leftarrow (C, (\alpha, \beta))$.

2. The Concept of the Strong Summability, Strong Convergence and Strong Semi-Convergences

For one-dimensional series the strong summability and strong convergence has been defined by Hyslop [5] and Tanovic–Miller [12]. Following their ways, the double series (1.1) is called summable $[C, (\alpha, \beta)]_\lambda$ to s, with index $\lambda > 0$ if in the cases of parameters

(i) $\alpha, \beta > 0$

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |\sigma_{i,k}^{(\alpha-1,\beta-1)} - s|^{\lambda} = o(1), \quad \text{as} \quad \min(m,n) \to \infty, \tag{2.1}$$

(ii) $\alpha = 0$, $\beta > 0$ or $\alpha > 0$, $\beta = 0$ or $\alpha = \beta = 0$,

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |\Delta_{11}((i+1)(k+1)\sigma_{i,k}^{(\alpha,\beta)}) - s|^{\lambda} = o(1), \quad \text{as} \quad \min(m,n) \to \infty, \tag{2.2}$$

where $\Delta_{10}\omega_{i,k} = \omega_{i,k} - \omega_{i-1,k}$; $\Delta_{01}\omega_{i,k} = \omega_{i,k} - \omega_{i,k-1}$; $\Delta_{11}\omega_{i,k} = \omega_{i,k} - \omega_{i,k-1} - \omega_{i-1,k} + \omega_{i-1,k-1} = \Delta_{10}(\Delta_{01}\omega_{i,k}) = \Delta_{01}(\Delta_{10}\omega_{i,k})$, for $i, k = 0, 1, 2, \ldots$ and $\omega_{i,k} = 0$ if i or k or both are -1. Summabilities $[C, (0,0)]_{\lambda}$, $[C, (0,\beta)]_{\lambda}$, and $[C, (\alpha, 0)]_{\lambda}$ will be called strong convergence, strong semi-convergence in first parameter, and strong semi-convergence in second parameter, respectively. Using (1.3), the formulas (2.1) and (2.2) give back the concept of strong summability and strong convergence for one-dimensional series.

Considering (1.2) we use the following notations,

$$z_{m,n}^{(\alpha,\beta)} = m\,\Delta_{10}\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha-1)} A_{n-k}^{(\beta)} i a_{i,k}, \tag{2.3.1}$$

$$t_{m,n}^{(\alpha,\beta)} = n\,\Delta_{01}\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha)} A_{n-k}^{(\beta-1)} k a_{i,k}, \tag{2.3.2}$$

$$\tau_{m,n}^{(\alpha,\beta)} = mn\,\Delta_{11}\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha-1)} A_{n-k}^{(\beta-1)} ik a_{i,k}, \tag{2.4}$$

where the last expression is the (m,n)th $(C, (\alpha, \beta))$ mean of the sequence $\{mna_{m,n}\}_{m,n=0}^{\infty}$.

The strong summability has the following properties. The first one is obvious by Property 1.5 (see (1.5.3) with $\delta = \gamma = 1$ and $\bar{\alpha} = \bar{\beta} = 0$).

PROPERTY 2.5. *If* $\alpha, \beta > 0$ *then* $(C, (\alpha-1, \beta-1)) \Rightarrow [C, (\alpha, \beta)]_{\lambda}$ *for* $\lambda > 0$.

PROPERTY 2.6 [9, Theorem 3.2]. *If* $\lambda > 1$; $\alpha, \beta > 1/\lambda$ *and* $\delta, \gamma < 1 - 1/\lambda$ *then* $[C, (\alpha, \beta)]_{\lambda} \Rightarrow (C, (\alpha - \delta, \beta - \gamma))$.

PROPERTY 2.7 [9, Theorem 3.3]. $[C, (\alpha, \beta)]_1 \Rightarrow (C, (\alpha, \beta))$ *for* $\alpha, \beta \geqslant 0$.

PROPERTY 2.8 [9, Remark 2.3]. *If* $0<\lambda<1$ *and* $\alpha, \beta \geqslant 0$ *then* $[C, (\alpha, \beta)]_\lambda \not\Rightarrow (C, (\alpha, \beta))$.

PROPERTY 2.9 [9, Theorem 3.1]. $[C, (\alpha, \beta)]_\lambda \Rightarrow [C, (\alpha, \beta)]_\mu$ *for* $\alpha, \beta \geqslant 0$ *and* $\lambda > \mu > 0$.

PROPERTY 2.10 [9, Theorem 3.4]. *Let be* $\alpha, \beta \geqslant 0$ *and* $\lambda \geqslant 1$. *Then the series* (1.1) *is summable* $[C, (\alpha, \beta)]_\lambda$ *to* s *if and only if the following four conditions hold: series* (1.1) *is summable* $(C, (\alpha, \beta))$ *to* s,

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |z_{i,k}^{(\alpha,\beta)}|_\lambda = o(1), \qquad as \quad \min(m,n) \to \infty, \tag{2.10.1}$$

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |t_{i,k}^{(\alpha,\beta)}|^\lambda = o(1), \qquad as \quad \min(m,n) \to \infty, \tag{2.10.2}$$

and

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |\tau_{m,n}^{(\alpha,\beta)}|^\lambda = o(1), \qquad as \quad \min(m,n) \to \infty. \tag{2.10.3}$$

PROPERTY 2.11 [9, Remark 3.7]. *If* $\lambda \geqslant 1$, $\lambda \geqslant \mu > 0$ *and* $\alpha, \beta. \gamma, \delta \geqslant 0$ *then* $[C, (\alpha, \beta)]_\lambda \Rightarrow [C, (\alpha+\delta, \beta+\gamma)]_\mu$.

PROPERTY 2.12 [10, Theorem].

(i) *If* $\mu > \lambda > 1$ *and* $\alpha, \beta \geqslant 0$; $\delta, \gamma \geqslant 1/\lambda - 1/\mu$ *then* $[C, (\alpha, \beta)]_\lambda \Rightarrow [C, (\alpha+\delta, \beta+\gamma)]_\mu$.

(ii) *If* $\mu > \lambda = 1$ *and* $\alpha, \beta > 0$; $\delta, \gamma > 1 - 1/\mu$ *then* $[C, (\alpha, \beta)]_1 \Rightarrow [C, (\alpha+\delta, \beta+\gamma)]_\mu$.

PROPERTY 2.13 [9, Theorem 3.5]. *Let* $\lambda \geqslant 1$ *and* $\alpha, \beta \geqslant 0$. *Then the series* (1.1) *is summable* $[C, (\alpha, \beta)]_\lambda$ *to* s *if and only if the condition* (2.2) *fulfils.*

PROPERTY 2.14. *If one of the cases*

(i) $\lambda = 1$ *and* $\alpha = \beta = 0$

(ii) $\lambda > 1$ *and* $0 \leqslant \alpha, \beta < 1 - 1/\lambda$

is fulfilled then $[C, (\alpha, \beta)]_\lambda \Rightarrow (C, (0, 0))$.

Proof of Property 2.14. In the case (i) we get the statement from the case $\alpha = \beta = 0$ of Property 2.7, so we may assume that $\lambda > 1$ and $0 \leqslant \alpha, \beta < 1 - 1/\lambda$. In the next we consider the cases

(a) $1<\lambda\leqslant 2$ and $0\leqslant\alpha, \beta<1-1/\lambda$,

(b) $\lambda>2$ and $0\leqslant\alpha, \beta\leqslant 1/\lambda$,

(c) $\lambda>2$ and $0\leqslant\alpha\leqslant 1/\lambda$, $1/\lambda<\beta<1-1/\lambda$,

(d) $\lambda>2$ and $1/\lambda<\alpha<1-1/\lambda$, $0<\beta\leqslant 1/\lambda$,

(e) $\lambda>2$ and $1/\lambda<\alpha, \beta<1-1/\lambda$.

Ad (a). Let us consider a positive number η, such that

$$\eta>\max\left(\frac{1-\alpha\lambda}{\alpha+1/\lambda}, \frac{1-\beta\lambda}{\beta+1/\lambda}\right).$$

Applying the Property 2.12 (i), in the case of indices $\lambda+\eta, \lambda$ and parameters α, β with $\delta_1=\gamma_1=1/\lambda-1/(\lambda+\eta)$, the summability $[C, (\alpha+1/\lambda-1/(\lambda+\eta), \beta+1/\lambda-1/(\lambda+\eta))]_{\lambda+\eta}$ of series (1.1) is obtained. Having that

$$\alpha+\frac{1}{\lambda}-\frac{1}{\lambda+\eta}>\frac{1}{\lambda+\eta}, \qquad \beta+\frac{1}{\lambda}-\frac{1}{\lambda+\eta}>\frac{1}{\lambda+\eta}$$

and choosing $\delta_2=\alpha+1/\lambda-1/(\lambda+\eta)$ $(<1-1/(\lambda+\eta))$ and $\gamma_2=\beta+1/\lambda-1/(\lambda+\eta)$ $(<1-1/(\lambda+\eta))$ we may apply Property 2.6 in the case of index $\lambda+\eta$ and parameters $\alpha+1/\lambda-1/(\lambda+\eta)$, $\beta+1/\lambda-1/(\lambda+\eta)$.

Ad (b). Let us choose the numbers δ_1 and γ_1 such that

$$\frac{1}{\lambda}-\alpha<\delta_1<1-\frac{1}{\lambda}-\alpha, \qquad \frac{1}{\lambda}-\beta<\gamma_1<1-\frac{1}{\lambda}-\beta$$

and apply Property 2.11 in the case of indices $\lambda=\mu$ and parameters α, β; then the summability $[C, (\alpha+\delta_1, \beta+\gamma_1)]_\lambda$ of series (1.1) is obtained. Having that $\alpha+\delta_1, \beta+\gamma_1>1/\lambda$ and choosing $\delta_2=\alpha+\delta_1$, $\gamma_2=\beta+\gamma_1$ we may apply Property 2.6 in the case of index λ and parameters $\alpha+\delta_1, \beta+\gamma_1$.

Ad (c). Choosing $1/\lambda-\alpha<\delta_1<1-1/\lambda-\alpha$ and $\gamma_1=0$ we apply Property 2.11 in the case of indices $\lambda=\mu$ and parameters α, β; then the summability $[C, (\alpha+\delta_1, \beta)]_\lambda$ of series (1.1) is obtained. Having that $\alpha+\delta_1, \beta>1/\lambda$ and choosing $\delta_2=\alpha+\delta_1$ and $\gamma_2=\beta$, we may apply Property 2.6 in the case of index λ and parameters $\alpha+\delta_1, \beta$.

Ad (d). This case is similar to the case (c), so we omit its proof.

Ad (e). Choosing $\delta=\alpha$ and $\gamma=\beta$ we get the convergence in Pringsheim's sense with a bound of the series (1.1) by the Property 2.6, immediately.

Remark 2.15. Let be α, β, λ given positive numbers and let us consider a series (1.1) with the following $(C, (\alpha-1, \beta-1))$ means

$$\sigma_{m,n}^{(\alpha-1,\beta-1)}=\begin{cases} m^{1/\lambda} & \text{if } \ n=0 \text{ and } m=0,1,2,..., \\ n^{1/\lambda} & \text{if } \ m=0 \text{ and } n=0,1,2,..., \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this series is summable $C_{\alpha-1,\beta-1}$ to $s=0$, but the sequence

$$\left\{\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}|\sigma_{i,k}^{(\alpha-1,\beta-1)}|^{\lambda}\right\}_{m,n=0}^{\infty}$$

does not converge in Pringsheim's sense, so we can see that *using the concept of* $C_{\alpha,\beta}$*-summability, the Property* 2.5 *does not remain valid.*

3. Results for the Double Orthogonal Series

Let $(X, \mathscr{F}, \rho)$ be a given arbitrary positive measure space and $\{\varphi_{i,k}(x)\}_{i,k=0}^{\infty}$ an orthonormal system on X. We consider the double orthogonal series

$$\sum_{i,k=0}^{\infty} c_{i,k}\varphi_{i,k}(x). \tag{3.1}$$

For series (3.1) we use $\sigma_{m,n}^{(\alpha,\beta)}(x)$, $z_{m,n}^{(\alpha,\beta)}(x)$, $t_{m,n}^{(\alpha,\beta)}(x)$, and $\tau_{m,n}^{(\alpha,\beta)}(x)$ in the same sense as under (1.2), (2.3.1), (2.3.2), and (2.4) with $a_{i,k}=c_{i,k}\varphi_{i,k}(x)$.

By the well-known Riesz–Fischer theorem, if

$$\sum_{i,k=0}^{\infty} c_{i,k}^2<\infty, \tag{3.2}$$

then the series (3.1) is an orthogonal expansion, that is, there exists a function $f(x)\in L^2(X, \mathscr{F}, \rho)$, such that the rectangular partial sums

$$s_{m,n}(x)=\sum_{i=0}^{m}\sum_{k=0}^{n} c_{i,k}\varphi_{i,k}(x)$$

convergence to $f(x)$ in the L^2-metric, that is

$$\int (s_{m,n}(x)-f(x))^2\,d\rho(x)=o(1), \qquad \text{as} \quad \min(m,n)\to\infty.$$

Here and in the sequel, the integrals are taken over the entire space X. We call the function $f(x)$ to be the L^2-sum of the series (3.1).

In [11] we proved the following results.

THEOREM 3.3 [11, Theorem (i)]. *Let $0<v\leqslant\mu$, $\mu\geqslant 2$ and $\alpha, \beta>1-1/\mu$. If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log\log(i+k)]^2\,[\log\log(k+k)]^2<\infty \tag{3.3.1}$$

then the series (3.1) *is summable* $[C,(\alpha,\beta)]_v$ *a.e. to its* L^2*-sum.*

The special case $\mu=v=2$ of Theorem 3.3 had already been proved by Móricz [7, Theorem 6]. Using Theorem 3.3 and Property 2.6 with $\lambda=v$; $\alpha=1-1/\mu+\alpha^*/2$ $(>1/\mu)$, $\beta=1-1/\mu+\beta^*/2$; $\delta=1-1/\mu-\alpha^*/2$ and $\gamma=1-1/\mu-\beta^*/2$, where α^* and β^* are arbitrary positive numbers, we get

COROLLARY 3.4. *If the coefficients of orthogonal series* (3.1) *satisfy the condition* (3.3.1) *then it is summable* $(C,(\alpha^*,\beta^*))$ *a.e. to its* L^2*-sum for every* $\alpha^*, \beta^*>0$.

The Corollary 3.4—which in the case of single orthogonal series is the well-known Men'sov–Kaczmarz theorem—had been proved by Móricz [7, Theorem 1]. In the case $\alpha^*=\beta^*=1$, it was pointed out by Fedulov [4, Theorem 3], that Corollary 3.4 is exact in the sense that $\log\log t$ can not be replaced by any sequence $w(t)$ tending to ∞, slower as $t\to\infty$. Hence, considering Property 2.10, we can see that Theorem 3.3 is the best possible in the same sense as Corollary 3.4 is.

THEOREM 3.5 [11, Theorem (ii)]. *Let $0<v\leqslant\mu$, $\mu\geqslant 2$, $\alpha=1-1/\mu$ and $\beta>1-1/\mu$. If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(i+2)][\log\log(k+4)]^2<\infty$$

then the series (3.1) *is summable* $[C,(\alpha,\beta)]_v$ *a.e. to its* L^2*-sum.*

THEOREM 3.6 [11, Theorem (iv)]. *Let $0<v\leqslant\mu$, $\mu\geqslant 2$ and $\alpha=\beta=1-1/\mu$. If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(i+2)][\log(k+2)]<\infty$$

then the series (3.1) *is summable* $[C,(\alpha,\beta)]_v$ *a.e. to its* L^2*-sum.*

For the cases in which one of parameters is smaller than $1-1/\mu(\mu \geqslant 2)$, we prove the following theorems.

THEOREM 3.7. *Let* $0<\nu \leqslant \mu$, $\mu \geqslant 2$, $0 \leqslant \alpha < 1-1/\mu$, *and* $\beta > 1-1/\mu$. *If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{2(1-1/\mu-\alpha)} [\log\log(k+4)]^2 < \infty \tag{3.7.1}$$

then the series (3.1) *is summable* $[C, (\alpha, \beta)]_\nu$ *a.e. to its* L^2*-sum.*

Let us assume that $-1<\alpha^*<0$ and $\beta^*>0$. Choosing the number

$$\mu > \frac{2}{1+\alpha^*},$$

then Theorem 3.7 and Property 2.6 with $\lambda=\mu$, $\alpha=1-1/\mu+\alpha^*$, $\beta=1-1/\mu+\beta^*/2$, $\gamma=1-1/\mu-\beta^*/2$ and $\delta=1-1/\mu-\eta$, where η is an arbitrarily small positive number, yield

COROLLARY 3.8. *If* $-1<\alpha^*<0$, $\beta^*>0$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{-2\alpha^*} [\log\log(k+4)]^2 < \infty \tag{3.8.1}$$

then the series (3.1) *is summable* $(C, \alpha^*+\eta, \beta^*)$ *a.e. to its* L^2*-sum, for any positive* η.

THEOREM 3.9. *Let* $0<\nu \leqslant \mu$, $\mu \geqslant 2$, $0 \leqslant \alpha < 1-1/\mu$ *and* $\beta = 1-1/\mu$. *If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{2(1-1/\mu-\alpha)} \log(k+2) < \infty \tag{3.9.1}$$

then the series (3.1) *is summable* $[C, (\alpha, \beta)]_\nu$ *a.e. to its* L^2*-sum.*

THEOREM 3.10. *Let* $0<\nu \leqslant \mu$, $\mu \geqslant 2$ *and* $0 \leqslant \alpha, \beta < 1-1/\mu$. *If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{1(1-1/\mu-\alpha)} (k+1)^{2(1-1/\mu-\beta)} < \infty \tag{3.10.1}$$

then the series (3.1) *is summable* $[C, (\alpha, \beta)]_\nu$ *a.e. to its* L^2*-sum.*

Let us assume that $-1<\alpha^*$, $\beta^*<0$. Choosing the number

$$\mu > \max\left(\frac{2}{1+\alpha^*}, \frac{2}{1+\beta^*}\right),$$

the Theorem 3.10 and Property 2.6 with $\lambda=\mu$, $\alpha=1-1/\mu+\alpha^*$, $\beta=1-1/\mu+\beta^*$, $\delta=1-1/\mu-\eta$ and $\gamma=1-1/\mu-\xi$, where η and ξ are arbitrarily small positive numbers, yield

COROLLARY 3.11. *If* $-1<\alpha^*, \beta^*<0$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2(i+1)^{-2\alpha^*}(k+1)^{-2\beta^*}<\infty \tag{3.11.1}$$

then the series (3.1) *is summable* $(C,(\alpha^*+\eta, \beta^*+\xi))$ *a.e. to its* L^2*-sum, for any positive* η *and* ξ.

The Corollary 3.11 is an extension of a result due to Kaczmarz [6] and Zygmund [13], from the one-dimensional orthogonal series to the two-dimensional one. We note that Sunouchi and Yano [8] proved that if in the case of one-dimensional orthogonal series $\sum_{k=0}^{\infty} c_k\varphi_k(x)$ the coefficients are satisfying the condition $\sum_{k=0}^{\infty} c_k^2(k+1)^{-2\alpha}<\infty$, $-1<\alpha<0$, then the orthogonal series is (C,α) summable a.e. This result indicates that Corollary 3.11 probably is not the best. The question whether Corollary 3.11 remains valid or not, when η or ξ or both are zero is open.

4. AUXILIARY RESULTS

Before starting the auxiliary results, we make the following convention. Given a double sequence $\{f_{m,n}(x)\}_{m,n=0}^{\infty}$ of functions we write

$$f_{m,n}(x)=o_x(1), \qquad \text{a.e., as } \min(m,n)\to\infty \text{ (or } m\to\infty, \text{ or } n\to\infty) \text{ or } \max(m,n)\to\infty$$

if $f_{m,n}(x)\to 0$ a.e. as $\min(m,n)\to\infty$ (or $\max(m,n)\to\infty$, or $m\to\infty$, or $n\to\infty$) and in addition there exists a function $F(x)\in L^2(X,\mathscr{F},\rho)$ such that

$$\sup_{m,n\geqslant 0}|f_{m,n}(x)|\leqslant F(x) \qquad \text{a.e.}$$

LEMMA 4.1 [1, Theorem 8.1]. *If*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(i+2)]^2\,[\log(k+2)]^2<\infty \tag{4.1.1}$$

then the series (3.1) *converges to its* L^2*-sum a.e. in Pringsheim's sense with a bound.*

LEMMA 4.2 [7, Theorem 2]. *If* $\alpha=0$, $\beta>0$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(i+2)]^2\,[\log\log(k+4)]^2<\infty \tag{4.2.1}$$

is fulfilled, then the series (3.1) *is summable* $(C,(\alpha,\beta))$ *to its* L^2*-sum a.e.*

LEMMA 4.3 [11, Lemma 1]. *The conditions in the cases*

(i) $\bar{\alpha}>\frac{1}{2}$

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(k+2)]^2<\infty, \tag{4.3.1}$$

(ii) $\bar{\alpha}=\frac{1}{2}$

$$\sum_{i,k=0}^{\infty} c_{i,k}^2[\log(i+2)][\log(k+2)]^2<\infty, \tag{4.3.2}$$

(iii) $-1<\bar{\alpha}<\frac{1}{2}$

$$\sum_{i,k=0}^{\infty} c_{i,k}^2(i+1)^{1-2\bar{\alpha}}\,[\log(k+2)]^2<\infty, \tag{4.3.3}$$

are sufficient for

$$\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2=o_x(1) \qquad \text{a.e. as } p\to\infty,$$

uniformly in n.

LEMMA 4.4 [11, Lemma 2 (iii)]. *Suppose that* $-1<\bar{\alpha}<\frac{1}{2}$, $\bar{\beta}>0$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2(i+1)^{1-2\bar{\alpha}}\,[\log\log(k+4)]^2<\infty. \tag{4.4.1}$$

Then

$$\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\;\sum_{n=2^q+1}^{2^{q+1}} |z_{m,n}^{(\bar{\alpha},\bar{\beta})}(x)|^2=o_x(1) \qquad \text{a.e. as } p\to\infty,$$

uniformly in q.

LEMMA 4.5. *Suppose that* $-1<\bar{\beta}<0$ *and that either*

(i) $\bar{\alpha}>\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^{2}(k+1)^{-2\bar{\beta}}<\infty \tag{4.5.1}$$

or

(ii) $\bar{\alpha}=\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^{2}[\log(i+2)](k+1)^{-2\bar{\beta}}<\infty \tag{4.5.2}$$

or

(iii) $-1<\bar{\alpha}<\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^{2}(i+1)^{1-2\bar{\alpha}}(k+1)^{-2\bar{\beta}}<\infty. \tag{4.5.3}$$

Then

$$\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\bar{\alpha},\bar{\beta})}(x)|^2=o_x(1) \qquad \text{a.e. as } p\to\infty, \tag{4.5.4}$$

uniformly in q.

Proof of Lemma 4.5. Writing α,β rather than $\bar{\alpha},\bar{\beta}$ and starting with (2.3.1) by the Minkowski's inequality we get

$$\begin{aligned}
&\left\{\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\alpha,\beta)}(x)|^2\right\}^{1/2}\\
&\leqslant\left\{\frac{1}{2^{p}}\sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{i=0}^{m}\sum_{k=0}^{2^q}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}ic_{i,k}\varphi_{i,k}(x)\right|^2\right\}^{1/2}\\
&\quad+\left\{\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\left|\sum_{i=0}^{m}\sum_{k=2^q+1}^{n}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}ic_{i,k}\varphi_{i,k}(x)\right|^2\right\}^{1/2}\\
&\quad+\left\{\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\left|\sum_{i=0}^{m}\sum_{k=0}^{n}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}\left(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\right)ic_{i,k}\varphi_{i,k}(x)\right|^2\right\}^{1/2}\\
&\equiv T_{p,q}^{(1)}(x)+T_{p,q}^{(2)}(x)+T_{p,q}^{(3)}(x).
\end{aligned}$$

In the first step we estimate $T_{p,q}^{(1)}$. Using

$$\rho_k=\begin{cases}-1 & \text{if } k=0\\ 0 & \text{if } k=1\\ 2^{k-2} & \text{if } k=2,3,\ldots\end{cases} \tag{4.5.5}$$

let us consider the sequence

$$c^*_{i,k}=\left\{\sum_{j=\rho_k+1}^{\rho_{k+1}} c^2_{i,j}\right\}^{1/2} \qquad (i,k=0,1,2,\dots)$$

and the system

$$\varphi^*_{i,k}(x)=\begin{cases}(1/c^*_{i,k})\sum_{j=\rho_k+1}^{\rho_{k+1}} c_{i,j}\varphi_{i,j}(x) & \text{if} \quad c^*_{i,k}\neq 0\\ \varphi_{i,\rho_{k+1}}(x) & \text{if} \quad c^*_{i,k}=0,\end{cases}$$

moreover, observe that

$$(T^{(1)}_{p,q}(x))^2=\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\left|\frac{1}{A^{(\alpha)}_m}\sum_{i=0}^{m}\sum_{k=0}^{q+1} A^{(\alpha-1)}_{m-i}\, i c^*_{i,k}\varphi^*_{i,k}(x)\right|^2.$$

Now the conditions (4.5.1), (4.5.2), and (4.5.3) yield that in the cases (i), (ii), and (iii) the series

$$\sum_{i,k=0}^{\infty}(c^*_{i,k})^2\,[\log(k+2)]^2,$$

$$\sum_{i,k=0}^{\infty}(c^*_{i,k})^2\,[\log(i+2)][\log(k+2)]^2,$$

and

$$\sum_{i,k=0}^{\infty}(c^*_{i,k})^2\,(i+1)^{1-2\alpha}\,[\log(k+2)]^2$$

are convergent, respectively.

Hence, considering the double orthogonal series

$$\sum_{i,k=0}^{\infty} c^*_{i,k}\varphi^*_{i,k}(x)$$

and casting a glance at (4.3.1)–(4.3.3), using again (2.3.1), we may apply Lemma 4.3 and get

$$\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\left|\frac{1}{A^{(\alpha)}_m}\sum_{i=0}^{m}\sum_{k=0}^{n} A^{(\alpha-1)}_{m-i}\, i c^*_{i,k}\varphi^*_{i,k}(x)\right|^2=o_x(1), \qquad \text{a.e. as } p\to\infty$$

uniformly in n. Writing $n=2^q$,

$$T^{(1)}_{p,q}(x)=o_x(1), \qquad \text{a.e. as } p\to\infty,$$

uniformly in q, holds.

In the second step we estimate $T^{(2)}_{p,q}$.

$$\sum_{p,q=0}^{\infty} \int |T^{(2)}_{p,q}(x)|^2 \, d\rho(x)$$
$$\leqslant K_\alpha \sum_{p,q=0}^{\infty} \frac{1}{2^{p+q}} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} \sum_{i=0}^{m} \sum_{k=2^q+1}^{n} (m-i+1)^{2\alpha-2}$$
$$\times (m+1)^{-2\alpha} i^2 c_{i,k}^2$$
$$\leqslant K_\alpha \sum_{m,q=0}^{\infty} \sum_{i=0}^{m} \sum_{k=2^q+1}^{2^{q+1}} (m-i+1)^{2\alpha-2} (m+1)^{-2\alpha-1} i^2 c_{i,k}^2$$
$$\leqslant K_\alpha \sum_{k,m=0}^{\infty} \sum_{i=0}^{m} (m-i+1)^{2\alpha-2} (m+1)^{-2\alpha-1} i^2 c_{i,k}^2$$
$$= K_\alpha \sum_{k,i=0}^{\infty} i^2 c_{i,k}^2 \sum_{m=i}^{\infty} (m-i+1)^{2\alpha-2} (m+1)^{-2\alpha-1}.$$

Considering

$$\sum_{m=i}^{\infty} (m-i+1)^{2\alpha-2} (m+1)^{-2\alpha-1} \leqslant \begin{cases} K_\alpha (i+1)^{-2} & \text{if } \alpha > \frac{1}{2} \\ K_\alpha (i+1)^{-2} \log(i+2) & \text{if } \alpha = \frac{1}{2} \\ K_\alpha (i+1)^{-2\alpha-1} & \text{if } -1 < \alpha < \frac{1}{2} \end{cases} \tag{4.5.6}$$

by conditions (4.5.1)–(4.5.3) we apply the Levi's theorem and get

$$T^{(2)}_{p,q}(x) = o_x(1) \qquad \text{a.e. as } \max(p, q) \to \infty.$$

In the third step we estimate $T^{(3)}_{p,q}$.

$$\sum_{p,q=0}^{\infty} \int |T^{(3)}_{p,q}(x)|^2 \, d\rho(x)$$
$$\leqslant K_\alpha \sum_{p,q=0}^{\infty} \frac{1}{2^{p+q}} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} \sum_{i=0}^{m} \sum_{k=0}^{n} \frac{(m-i+1)^{2\alpha-2}}{(m+1)^{2\alpha}}$$
$$\times \left(\frac{A^{(\beta)}_{n-k}}{A^{(\beta)}_n} - 1 \right)^2 i^2 c_{i,k}^2$$
$$\leqslant K_\alpha \sum_{m,n=0}^{\infty} \frac{1}{n+1} \frac{1}{(m+1)^{2\alpha+1}} \sum_{i=0}^{m} \sum_{k=0}^{n} (m-i+1)^{2\alpha-2} i^2$$
$$\times \left(\frac{A^{(\beta)}_{n-k}}{A^{(\beta)}_m} - 1 \right)^2 c_{i,k}^2$$
$$= K_\alpha \sum_{i,k=0}^{\infty} i^2 c_{i,k}^2 \left(\sum_{m=i}^{\infty} \frac{(m-i+1)^{2\alpha-2}}{(m+1)^{2\alpha+1}} \right) \left(\sum_{n=k}^{\infty} \frac{1}{n+1} \left(\frac{A^{(\beta)}_{n-k}}{A^{(\beta)}_n} - 1 \right)^2 \right).$$

Here we show that for any $k=0, 1, 2, \dots$

$$\sum_{n=k}^{\infty} \frac{1}{n+1}\left(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\right)^2=\begin{cases} O(1) & \text{if } -\frac{1}{2}<\beta<0 \\ O(\log(k+2)) & \text{if } \beta=-\frac{1}{2} \\ O((k+1)^{-2\beta-1}) & \text{if } -1<\beta<-\frac{1}{2} \end{cases} \qquad (*)$$

and this estimate is sharp.

Clearly, we may assume that $k \geqslant 1$. Using the familiar identity

$$A_m^{(\beta)}=\sum_{j=0}^{m} A_j^{(\beta-1)}$$

we can write

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{1}{n+1} \frac{1}{(A_n^{(\beta)})^2}(A_{n-k}^{(\beta)}-A_n^{(\beta)})^2 \\
&\quad=\sum_{n=k}^{\infty} \frac{1}{n+1} \frac{1}{(A_n^{(\beta)})^2}\left(\sum_{j=n-k+1}^{n} A_j^{(\beta-1)}\right)^2 \\
&\quad=O_\beta\left(\sum_{n=k}^{\infty} \frac{1}{(n+1)^{1+2\beta}}\left(\sum_{j=n-k+1}^{n} j^{\beta-1}\right)^2\right) \\
&\quad=O_\beta\left(\left(\sum_{n=k}^{2k-1}+\sum_{n=2k}^{\infty}\right) \frac{1}{(n+1)^{1+2\beta}}((n+1)^\beta-(n-k+1)^\beta)^2\right) \equiv I_1+I_2 .
\end{aligned}$$

In the case of I_1 we have $k \leqslant n<2k$ and obtain

$$I_1 \equiv O_\beta\left(\frac{1}{(k+1)^{2\beta+1}} \sum_{n=k}^{2k-1}((n+1)^\beta-(n-k+1)^\beta)^2\right).$$

It is easy to see that if $\beta<0$ then

$$\begin{aligned}
(1-2^\beta)(n-k+1)^\beta &\leqslant(n-k+1)^\beta-(n+1)^\beta \\
&\leqslant(n-k+1)^\beta \qquad (k \leqslant n \leqslant 2k-1)
\end{aligned}$$

and hence

$$\begin{aligned}
I_1&=O_\beta\left(\frac{1}{(k+1)^{2\beta+1}} \sum_{n=k}^{2k-1}(n-k+1)^{2\beta}\right)=O_\beta\left(\frac{1}{(k+1)^{2\beta+1}} \sum_{m=1}^{k} m^{2\beta}\right) \\
&=\begin{cases} O_\beta(1) & \text{if } -\frac{1}{2}<\beta<0 \\ O_\beta(\log k) & \text{if } \beta=-\frac{1}{2} \\ O_\beta((k+1)^{-2\beta-1}) & \text{if } \beta<-\frac{1}{2}. \end{cases}
\end{aligned}$$

In the case of I_2 we have $\frac{1}{2}(n+1) < n-k+1 < n+1$ and—by the Lagrange theorem—we obtain

$$((n+1)^\beta - (n-k+1)^\beta)^2 = O_\beta(k^2(n+1)^{2\beta-2}).$$

Hence,

$$I_2 = O_\beta\left(\sum_{n=2k}^{\infty} \frac{1}{(n+1)^{2\beta+1}} k^2(n+1)^{2\beta-2}\right) = O_\beta(1).$$

Collecting the estimates of I_1 and I_2 we obtain $(*)$.

Returning to the main line of the proof, it is sufficient to observe that $(*)$ yields the estimate

$$\sum_{n=k}^{\infty} \frac{1}{n+1}\left(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}} - 1\right)^2 \leqslant K_\beta(k+1)^{-2\beta} \qquad (-1<\beta<0;\ k=0, 1, 2, \ldots),$$

and using (4.5.6) again, by conditions (4.5.1)–(4.5.3) we apply the Levi's theorem and get

$$T_{p,q}^{(3)}(x) = o_x(1) \qquad \text{a.e. as } \max(p,q) \to \infty.$$

Collecting the estimations of $T_{p,q}^{(1)}$, $T_{p,q}^{(2)}$, and $T_{p,q}^{(3)}$, the estimation (4.5.4) is obtained.

Our next lemma is analogous to Lemma 4.5.

LEMMA 4.6. *Suppose that* $-1<\bar{\alpha}<0$ *and that either*

(i) $\bar{\beta} > \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{-2\bar{\alpha}} < \infty \tag{4.6.1}$$

or

(ii) $\bar{\beta} = \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{-2\bar{\alpha}} \log(k+2) < \infty \tag{4.6.2}$$

or

(iii) $-1 < \bar{\beta} < \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{-2\bar{\alpha}} (k+1)^{1-2\bar{\beta}} < \infty. \tag{4.6.3}$$

Then

$$\frac{1}{2^{p+q}} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} |t_{m,n}^{(\bar{\alpha},\bar{\beta})}(x)|^2 = o_x(1) \qquad \text{a.e. as } q \to \infty,$$

uniformly in p.

LEMMA 4.7 [11, Lemma 5(iii), (v), and (vi)]. *Suppose that* $-1 < \bar{\alpha} < \frac{1}{2}$ *and let one of the conditions*

(i) $\bar{\beta} > \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{1-2\bar{\alpha}} < \infty, \tag{4.7.1}$$

(ii) $\bar{\beta} = \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{1-2\bar{\alpha}} \log(k+2) < \infty, \tag{4.7.2}$$

(iii) $-1 < \bar{\beta} < \frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} c_{i,k}^2 (i+1)^{1-2\bar{\alpha}} (k+1)^{1-2\bar{\beta}} < \infty, \tag{4.7.3}$$

be satisfied. Then the series

$$\sum_{m,n=0}^{\infty} \frac{1}{(m+1)(n+1)} |\tau_{m,n}^{(\bar{\alpha},\bar{\beta})}(x)|^2$$

converges a.e.

LEMMA 4.8 [10, Lemma 8]. *Assume that* $\mu > \lambda > 1$, $\bar{\alpha}, \bar{\beta} > 1/\lambda - 1$, *and* $\delta = 1/\lambda - 1/\mu$. *If the conditions* (2.10.1)–(2.10.3) *are satisfied, then*

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha}+\delta,\, \bar{\beta}+\delta)}|^{\mu} = o(1) \qquad \text{as} \quad \min(M, N) \to \infty,$$

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |t_{m,n}^{(\bar{\alpha}+\delta,\, \bar{\beta}+\delta)}|^{\mu} = o(1) \qquad \text{as} \quad \min(M, N) \to \infty,$$

and

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |\tau_{m,n}^{(\bar{\alpha}+\delta,\bar{\beta}+\delta)}|^{\mu} = o(1) \qquad \text{as} \quad \min(M, N) \to \infty,$$

hold.

5. Proof of Theorems 3.7, 3.9, and 3.10

First, we mention that having the cases (iii), (v), and (vi) of the theorem in [11], we may assume that

$$\mu > 2$$

and have to prove our new theorems only in the following cases of parameters

(I) In the case of Theorem 3.7,

$$0 \leqslant \alpha < \frac{1}{2} - \frac{1}{\mu}; \qquad \beta > 1 - \frac{1}{\mu};$$

(II) In the case of Theorem 3.9,

$$0 \leqslant \alpha < \frac{1}{2} - \frac{1}{\mu}; \qquad \beta = 1 - \frac{1}{\mu};$$

(III) In the case of Theorem 3.10,

(a) $$0 \leqslant \alpha < \frac{1}{2} - \frac{1}{\mu}; \qquad \frac{1}{2} - \frac{1}{\mu} \leqslant \beta < 1 - \frac{1}{\mu}$$

(b) $$0 \leqslant \alpha < \frac{1}{2} - \frac{1}{\mu}; \qquad 0 \leqslant \beta < \frac{1}{2} - \frac{1}{\mu}.$$

Second, we remark that these theorems have the same proof-line, so we can make a common proof.

The proof is based on Property 2.10. We will show that the requirements

$$\text{the series (3.1) is summable } (C, (\alpha, \beta)) \text{ to its } L^2\text{-sum a.e.,} \tag{5.0.1}$$

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\alpha,\beta)}(x)|^{\mu} = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty, \tag{5.0.2}$$

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |t_{m,n}^{(\alpha,\beta)}(x)|^{\mu} = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty, \tag{5.0.3}$$

and

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}|\tau_{m,n}^{(\alpha,\beta)}(x)|^{\mu}=o_x(1) \qquad \text{a.e. as } \min(M,N)\to\infty, \tag{5.0.4}$$

are fulfilled, in each of the cases I–III.

Observing that each of conditions (3.7.1), (3.9.1), and (3.10.1) implies (4.2.1), moreover (3.10.1) implies (4.1.1), by Lemmas 4.1 and 4.2 moreover by Property 1.7, the requirement (5.0.1) is obtained.

In the sequel let be

$$\bar{\alpha}=\alpha-\frac{1}{2}+\frac{1}{\mu} \qquad \text{and} \qquad \bar{\beta}=\beta-\frac{1}{2}+\frac{1}{\mu}. \tag{5.1}$$

Now we write the cases I–III, in the following form:

$$\text{(I)} \qquad -\frac{1}{2}+\frac{1}{\mu}\leqslant\bar{\alpha}<0; \qquad \bar{\beta}>\frac{1}{2}$$

$$\text{(II)} \qquad -\frac{1}{2}+\frac{1}{\mu}\leqslant\bar{\alpha}<0; \qquad \bar{\beta}=\frac{1}{2}$$

$$\text{(III)} \quad \text{(a)} \qquad -\frac{1}{2}+\frac{1}{\mu}\leqslant\bar{\alpha}<0; \qquad 0\leqslant\bar{\beta}<\frac{1}{2}$$

$$\text{(b)} \qquad -\frac{1}{2}+\frac{1}{\mu}\leqslant\bar{\alpha}<0; \qquad -\frac{1}{2}+\frac{1}{\mu}\leqslant\bar{\beta}<0.$$

Observing that (3.7.1) $\Leftrightarrow$ (4.4.1), (3.9.1) $\Rightarrow$ (4.4.1), and (3.10.1) $\Rightarrow$ ((4.3.3), (4.4.1), and (4.5.3)), the Lemmas 4.3, 4.4, and 4.5 yield that

$$\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}|z_{m,n}^{(\bar{\alpha},0)}(x)|^2=o_x(1) \qquad \text{a.e. as } p\to\infty \tag{5.2}$$

uniformly in n, and

$$\frac{1}{2^{p+q}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\bar{\alpha},\bar{\beta})}(x)|^2=o_x(1) \qquad \text{a.e. as } p\to\infty \tag{5.3}$$

uniformly in q.

Let us choose the integer number w, such that $2^{w-1}<M\leqslant 2^w$. Using (4.5.5), we can write for any $M=2, 3, \ldots$ and $n=0, 1, 2, \ldots$, that

$$\frac{1}{M+1}\sum_{m=0}^{M}|z_{m,n}^{(\bar{\alpha},0)}(x)|^2\leqslant\frac{2}{2^w+1}\sum_{p=0}^{w+1}2^p\frac{1}{2^p}\sum_{m=\rho_p+1}^{\rho_{p+1}}|z_{m,n}^{(\bar{\alpha},0)}(x)|^2.$$

The estimate (5.2) means that there is a function $F_1 \in L^2(X, \mathscr{F}, \rho)$, such that for any $p=0, 1, 2, \ldots$ and $n=0, 1, 2, \ldots$,

$$\frac{1}{2^p} \sum_{m=\rho_p+1}^{\rho_{p+1}} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 \leqslant F_1(x) \qquad \text{a.e.},$$

furthermore for any positive ε, there is a number $\kappa=\kappa_x(\varepsilon)$, such that if $p>\kappa$ then the inequality

$$\frac{1}{2^p} \sum_{m=\rho_p+1}^{\rho_{p+1}} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 < \varepsilon \qquad \text{a.e.},$$

holds for every $n=0, 1, 2, \ldots$.

Hence, we get that for any $M=2, 3, \ldots$ and $n=0, 1, 2, \ldots$

$$\frac{1}{M+1} \sum_{m=0}^{m} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 \leqslant 8F_1(x) \qquad \text{a.e.}$$

furthermore, assuming that $M>2^{\kappa+1}$ we can write

$$\frac{1}{M+1} \sum_{m=0}^{M} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 \leqslant \frac{2}{2^w+1} \left(\sum_{p=0}^{\kappa} + \sum_{p=\kappa+1}^{w+1} \right) 2^p \frac{1}{2^p} \sum_{m=\rho_p+1}^{\rho_{p+1}} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2,$$

so the right hand side is small if M is large enough, a.e., uniformly in n. Of course these yield

$$\frac{1}{M+1} \sum_{m=0}^{M} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 = o_x(1) \qquad \text{a.e. as } \min(M, n) \to \infty.$$

Using Property 1.5 (see (1.5.2) with $\beta=0$ and $\gamma=1$), we obtain

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha},0)}(x)|^2 = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty. \tag{5.4}$$

Choosing the integer numbers w and v, such that $2^{w-1} < M \leqslant 2^w$ and $2^{v-1} < N \leqslant 2^v$, furthermore using (4.5.5), we can write for any $M=2, 3, \ldots$ and $N=2, 3, \ldots$ that

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2$$

$$\leqslant \frac{4}{(2^w+1)(2^v+1)} \sum_{p=0}^{w+1} \sum_{q=0}^{v+1} 2^{p+q} \frac{1}{2^p 2^q} \sum_{m=\rho_p+1}^{\rho_{p+1}} \sum_{n=\rho_q+1}^{\rho_{q+1}} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2.$$

The estimate (5.3) means that there is a function $F_2 \in L^2(X, \mathscr{F}, \rho)$, such that for any $p=0, 1, 2, \ldots$ and $q=0, 1, 2, \ldots$.

$$\frac{1}{2^p 2^q} \sum_{m=\rho_p+1}^{\rho_{p+1}} \sum_{n=\rho_q+1}^{\rho_{q+1}} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 \leqslant F_2(x) \qquad \text{a.e.,}$$

furthermore, for any positive ε, there is a number $\kappa=\kappa_x(\varepsilon)$ such that if $p>\kappa$, then for any $q=0, 1, 2, \ldots$, the inequality

$$\frac{1}{2^p 2^q} \sum_{m=\rho_p+1}^{\rho_{p+1}} \sum_{n=\rho_q+1}^{\rho_{q+1}} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 < \varepsilon \qquad \text{a.e.,}$$

holds.

Hence we get that for any $M=2, 3, \ldots$ and $N=2, 3, \ldots$,

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 \leqslant 64F_2(x) \qquad \text{a.e.}$$

Furthermore assuming that $M>2^{\kappa+1}$ we can write

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2$$

$$\leqslant \frac{4F_2(x)}{(2^w+1)(2^v+1)} \sum_{p=0}^{\kappa} \sum_{q=0}^{v+1} 2^{p+q} + \frac{4\varepsilon}{(2^w+1)(2^v+1)} \sum_{p=\kappa+1}^{w+1} \sum_{q=0}^{v+1} 2^{p+q}$$

$$< \frac{2^{\kappa+5}F_2(x)}{M} + 64\varepsilon \qquad (N=0, 1, 2, \ldots) \qquad \text{a.e.}$$

Combining this with (5.4) we have

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |z_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty, \tag{5.5}$$

in each of the cases I–III.

Observing that $(3.7.1) \Rightarrow (4.6.1)$, $(3.9.1) \Rightarrow (4.6.2)$, and $(3.10.1) \Rightarrow (4.6.3)$ the Lemma 4.6 yields that

$$\frac{1}{2^{p+q}} \sum_{m=\rho_p+1}^{\rho_{p+1}} \sum_{n=\rho_q+1}^{\rho_{q+1}} |t_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 = o_x(1) \qquad \text{a.e. as } q \to \infty,$$

uniformly in p. Hence, by a similar computation to the former one, we obtain

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |t_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty, \tag{5.6}$$

in each of the cases I–III.

Observing that (3.7.1) ⇒ (4.7.1), (3.9.1) ⇔ (4.7.2), and (3.10.1) ⇔ (4.7.3), the Lemma 4.7 yields that the series (4.7.4) converges a.e. Hence, considering Lemma 3.3 in [2] with

$$a_{m,n} = \frac{1}{(m+1)(n+1)} |\tau_{m,n}^{(\bar{\alpha},\beta)}(x)|^2,$$

we can conclude that

$$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} |\tau_{m,n}^{(\bar{\alpha},\beta)}(x)|^2 = o_x(1) \qquad \text{a.e. as } \min(M, N) \to \infty, \tag{5.7}$$

in each of the cases I–III.

Choosing

$$\delta = \frac{1}{2} - \frac{1}{\mu}$$

and casting a glance at (5.1), by (5.5), (5.6) and (5.7) we may apply Lemma 4.8, with $\lambda = 2$, the requirements (5.0.2), (5.0.3), and (5.0.4) are fulfilled.

Finally, applying Property 2.9, our proof is complete.

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References

1. R. P. Agnew, On double orthogonal series, *Proc. London Math. Soc.* **33** (1932), 420–434.
2. J. M. Ash and G. V. Welland, Convergence, uniqueness, and summability of multiple trigonometric series, *Trans. Amer. Math. Soc.* **163** (1972), 401–436.
3. V. G. Chelidze, Some methods of summability of double series and double integrals, *Tbilisi*, 1977. [in Russian]

4. V. S. FEDULOV, On $(C, 1, 1)$-summability of double orthogonal series, *Ukrain Mat. Zh.* **7** (1955), 433–442. [in Russian]
5. J. M. HYSLOP, Note on the strong summability of series, *Proc. Glasgow Math. Assoc.* **1** (1952), 16–20.
6. S. KACZMARZ, Über die Konvergenz der Reihen von Orthogonalfunktionen, *Math. Z.* **23** (1925), 263–270.
7. F. MÓRICZ, On the $(C, \alpha \geqslant 0, \beta \geqslant 0)$-summability of double orthogonal series, *Stud. Math.* **81** (1985), 79–94.
8. G. SUNOUCHI AND S. YANO, Convergence and summability of orthogonal series, *Proc. Acad. Japan* **26** (1950), 10–16.
9. I. SZALAY, On the strong Cesaro summability of double series, submitted for publication.
10. I. SZALAY, A high-indices theorem for strong Cesaro summability of double series, submitted for publication.
11. I. SZALAY, On the strong Cesaro summability of double orthogonal series, *Stud. Math.* **91** (1988), 5–23.
12. N. TANOVIC-MILLER, On strong convergence of trigonometric and Fourier series, *Acta Math. Hungar.* **42** (1983), 35–43.
13. A. ZYGMUND, Une remarque sur un theoreme de M. Kaczmarz, *Math. Z.* **25** (1926), 297–298.